Torsion Testing for Shear Modulus of Thin Orthotropic Sheet

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Introduction

ORTHOTROPIC materials have nine independent elastic constants, of which three are shear moduli for three orthogonal planes. The simpler transversely isotropic materials have five distinct constants, of which only one is a shear constant, the remaining shear properties being determined by the uniaxial moduli. In each of these material classes shear tests are required to evaluate completely the elastic constants, and methods are needed for extracting these constants from test data.

Although a procedure is available for obtaining all the elastic constants of orthotropic materials from a series of uniaxial tensile or compression tests alone[1] such a method requires that the tests be made in a wide variety of orientations. For thin sheet this is not possible, but torsion tests provide a suitable alternative.

Although an analysis of orthotropic torsion has existed since the last century no practical use has been made of it, with the significant exception of the work of Trayer and March[2] who studied the behavior of wood.

This Note demonstrates a method of shear testing and of data reduction based on the torsion analysis which can be used for obtaining the elastic shear constants of thin orthotropic sheets. The method has been applied to a study of a cross-rolled beryllium sheet.

Torsion of Thin Orthotropic Sheet

Torsion tests on rectangular strips of thin sheet were recently proposed by Dai[3] as a means of obtaining the shear moduli of beryllium sheet. Although the rigorous analysis of a rectangular cross section of orthotropic material has long been available [Love[4] attributes its use to Voigt[5]; also see Sokolnikoff[6]] the resulting formula for stiffness is of no immediate use for evaluating properties from experimental data, since two moduli are inextricably involved in it. Only for very slender sections does the formula become approximately linear in one of these moduli, enabling simple data reduction procedures to be used. A practicable method for obtaining the moduli can be developed through a feature of the torsion analysis brought out by Love[1] in a discussion of isotropic torsion: the analysis is generally made in terms of trigonometric functions of the thickness variable, but it could equally well be made starting from the width variable. In that case, the resulting formula would be of identical form, but with the appropriate symbols for thickness and width, and for the corresponding moduli, interchanged. Thus, the slender approximation can be used either for strips of very narrow, or of very wide cross sections.

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Consider a strip in torsion about an axis x which coincides with one of the principal directions of the sheet. The transverse axis y and the thickness axis z of the strip coincide with the other principal axes of the sheet. The thickness and width of the strip are t and w, respectively. The shear modulus for deformation in the plane of the sheet is G_{xy} whereas that in a plane containing the torsion axis and the thickness direction is G_{xz} and that in a plane containing the transverse and thickness directions is G_{yz}.

According to Love[4] the torsional stiffness of the strip is given by the formulas

$$K_x = (32/\pi^4)wt^3G_{xy}S(N) = (32/\pi^4)tw^3G_{xz}S(1/N) \quad (1)$$

in which the torsion constant S is a function of the effective thickness-width ratio defined as

$$N = (G_{xy}/G_{xz})^{1/2}(t/w) \quad (2)$$

and is given by the series

$$S(N) = \sum_{n=1,3\ldots}^{\infty} \frac{1}{n^4}\left[1 - \frac{\tanh(\pi/2)n/N}{(\pi/2)n/N}\right] \quad (3)$$

This torsion constant is proportional to that defined as $k(b/a)$ by Timoshenko and Goodier[7] for the torsion of isotropic strips. When they are evaluated for the same value of their respective arguments, N and b/a, the two torsion constants have values related by the formula

$$S = (32/\pi^4)k \quad (4)$$

Equivalence of the two alternative forms of the torsional stiffness formula Eq. (1) is assured because the torsion constant satisfies the following identity:

$$S(1/N) = N^2S(N) \quad (5)$$

When N is very small, or very large so that $1/N$ is very small (say $N < \pi/4$ or $1/N < \pi/4$, i.e., $N > 4/\pi$, respectively), the torsional constant then has the approximate forms

$$S(N) \simeq \sum_{n=1,3\ldots}^{\infty}\left[\frac{1}{n^4} - \frac{2N}{\pi}\frac{1}{n^5}\right] = \frac{\pi^4}{96}(1 - 0.6302482N) \quad (6a)$$

or

$$S\left(\frac{1}{N}\right) \simeq \sum_{n=1,3\ldots}^{\infty}\left[\frac{1}{n^4} - \frac{2/N}{\pi}\frac{1}{n^5}\right] = \frac{\pi^4}{96}\left(\frac{1 - 0.6302}{N}\right) \quad (6b)$$

respectively. The sum of the odd inverse fifth powers cannot be given in terms of known numbers[8] but by computation to 56 terms the series is found to be

$$\sum_{n=1,3\ldots}^{56}\left(\frac{1}{n}\right)^5 = 1.0045237604$$

Use of these approximations in the stiffness formulas of Eq. (1) allows them to be written in the simplified forms below:

$$K_{red} = (3/wt^3)K_x = G_{xy} - 0.63025G_{xy}(G_{xy}/G_{xz})^{1/2}(t/w) \quad (7a)$$

or

$$(t/w)^2K_{red} = G_{xz} - 0.6302G_{xz}(G_{xz}/G_{xy})^{1/2}(w/t) \quad (7b)$$

respectively. These formulas are linear in the thickness-width ratio t/w, or its inverse w/t, a feature that can be exploited by performing tests on strips of various aspect ratios. Use of the formulas is restricted by the mathematical condition that the thickness to width ratio be small, or large, relative to the anisotropy ratio according to the conditions

$$(t/w) < (\pi/4)(G_{xz}/G_{xy})^{1/2} \quad (8a)$$

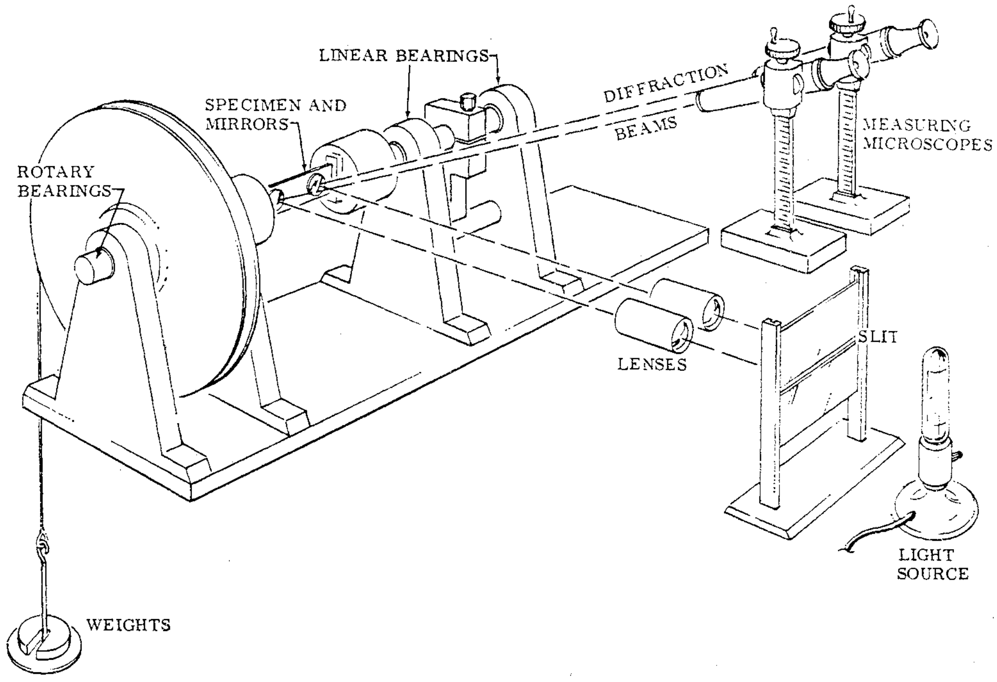

Fig. 1 Schematic of torsion test equipment.

or

$$(w/t) < (\pi/4)(G_{xy}/G_{xz})^{1/2} \quad \text{(8b)}$$

respectively. Moduli obtained by use of formulas of Eqs. (7) should be applied to the restrictions of Eqs. (8) to verify their validity.

The practical limitation that wrinkling would arise with very thin sheet, say with $(t/w) < \frac{1}{20}$, imposes an additional restriction on the cross sections to which these formula can be applied. In particular, when the anisotropy is such that the thickness shear modulus is very much less than the in-plane modulus, e.g., $G_{xz} < G_{xy}/400$, then the restriction of Eq. (8a) reaches the wrinkling limit, so that Eq. (7a) is of no value. For such materials Eq. (7b) can be used, with sections which approach a square.

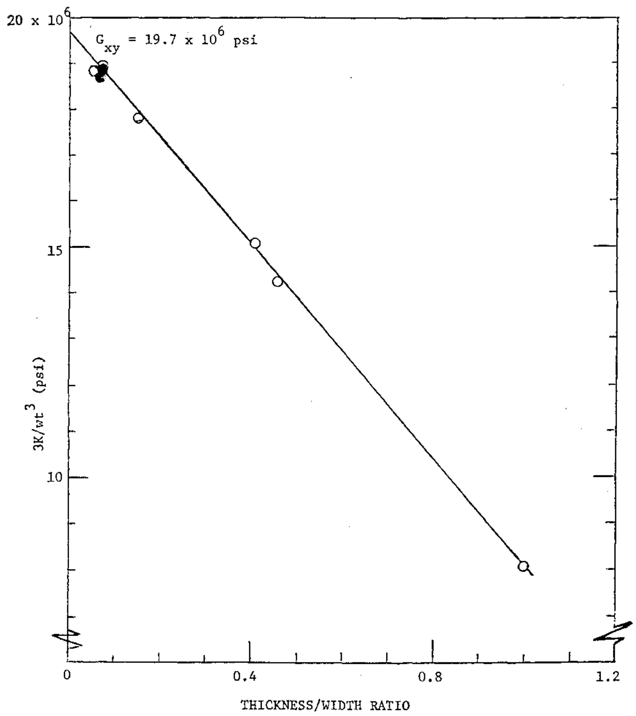

Fig. 2 Reduced torsional stiffness data as a function of thickness-width ratio (S-200 Be sheet).

Torsion Testing

Torsion tests were conducted on cross-rolled S-200 beryllium sheet specimens 0.075 in. thick and 8 in. long (using a 4 in. gage length) with widths of 0.07 in., 0.50 in., and 1.00 in. Torque was applied using dead weight as depicted in the schematic drawing of the torsion equipment shown in Fig. 1. Mirrors attached to either end of the specimen gage length were used to measure the twist of the specimen using travelling microscopes.

The reduced torsional stiffness defined in Eq. (7a) is plotted versus the thickness-width ratio in Fig. 2.

The intercept at zero t/w of a straight line drawn through the data points is the shear modulus in the plane of the specimens, G_{xy}; the negative of the slope, K', of the line provides the shear modulus in a through-thickness plane oriented along the axis, according to the formula obtained from Eq. (7a):

$$G_{yz} = 0.3972\ G_{xy}^3/K'^2 \quad (9)$$

Evidently this modulus can only be determined accurately when the straight line is well defined, since the factor G_{xy}^3 triples any error in the intercept, while error in the slope is doubled in the factor K'^2.

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